

## ON THE VOLUME OF DOUBLE TWIST LINK CONE-MANIFOLDS

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ABSTRACT. We consider the double twist link  $J(2m+1, 2n+1)$  which is the two-bridge link corresponding to the continued fraction  $(2m+1) - 1/(2n+1)$ . It is known that  $J(2m+1, 2n+1)$  has reducible nonabelian  $SL_2(\mathbb{C})$ -character variety if and only if  $m = n$ . In this paper we give a formula for the volume of hyperbolic cone-manifolds of  $J(2m+1, 2m+1)$ . We also give a formula for the A-polynomial 2-tuple corresponding to the canonical component of the character variety of  $J(2m+1, 2m+1)$ .

## 1. INTRODUCTION

For a hyperbolic link  $\mathcal{L}$  in  $S^3$ , let  $E_{\mathcal{L}} = S^3 \setminus \mathcal{L}$  be the link exterior and let  $\rho_{\text{hol}}$  be a holonomy representation of  $\pi_1(E_{\mathcal{L}})$  into  $PSL_2(\mathbb{C})$ . Thurston [Th] showed that  $\rho_{\text{hol}}$  can be deformed into an  $\ell$ -parameter family  $\{\rho_{\alpha_1, \dots, \alpha_\ell}\}$  of representations to give a corresponding family  $\{E_{\mathcal{L}}(\alpha_1, \dots, \alpha_\ell)\}$  of singular complete hyperbolic manifolds, where  $\ell$  is the number of components of  $\mathcal{L}$ . In this paper we consider only the case where all of  $\alpha_j$ 's are equal to a single parameter  $\alpha$ . In which case we also denote  $E_{\mathcal{L}}(\alpha_1, \dots, \alpha_\ell)$  by  $E_{\mathcal{L}}(\alpha)$ . These  $\alpha$ 's and  $E_{\mathcal{L}}(\alpha)$ 's are called the cone-angles and hyperbolic cone-manifolds of  $\mathcal{L}$ , respectively. We consider the complete hyperbolic structure on a link complement as the cone-manifold structure with cone-angle zero. It is known that for a two-bridge link  $\mathcal{L}$  there exists an angle  $\alpha_{\mathcal{L}} \in [\frac{2\pi}{3}, \pi)$  such that  $E_{\mathcal{L}}(\alpha)$  is hyperbolic for  $\alpha \in (0, \alpha_{\mathcal{L}})$ , Euclidean for  $\alpha = \alpha_{\mathcal{L}}$ , and spherical for  $\alpha \in (\alpha_{\mathcal{L}}, \pi)$  [HLM, Ko1, Po, PW]. A method for computing the volume of hyperbolic cone-manifolds of links was outlined in [HLM], and explicit volume formulas have been known for hyperbolic cone-manifolds of the links  $5_1^2$ ,  $6_2^2$ ,  $6_3^2$ ,  $7_3^2$  (see [HLMR] and references therein) and of twisted Whitehead links [Tr].

For integers  $m$  and  $n$ , consider the double twist link  $J(2m+1, 2n+1)$  which is the two-bridge link corresponding to the continued fraction  $(2m+1) - 1/(2n+1)$  (see Figure 1). It was shown by Petersen and the author [PT] that  $J(2m+1, 2n+1)$  has reducible nonabelian  $SL_2(\mathbb{C})$ -character variety if and only if  $m = n$ . In this paper we are interested in the double twist link  $\mathcal{L}_m = J(2m+1, 2m+1)$ , since the canonical component of the character variety of  $\mathcal{L}_m$  has a rather nice form (see Remark 3.4). Here a canonical component of the character variety of a hyperbolic link  $\mathcal{L}$  is a component containing the character of a lift of a holonomy representation of  $\pi_1(E_{\mathcal{L}})$  to  $SL_2(\mathbb{C})$ .

Let  $\{S_j(v)\}_{j \in \mathbb{Z}}$  be the sequence of Chebychev polynomials of the second kind defined by  $S_0(v) = 1$ ,  $S_1(v) = v$  and  $S_j(v) = vS_{j-1}(v) - S_{j-2}(v)$  for all integers  $j$ . Let

$$R_{\mathcal{L}_m}(s, z) = (s^2 + s^{-2} + 2 - z)(S_m^2(z) + S_{m-1}^2(z)) - 2(s^2 + s^{-2})S_m(z)S_{m-1}(z) - z.$$

The volume of the hyperbolic cone-manifold of  $\mathcal{L}_m$  is computed as follows.

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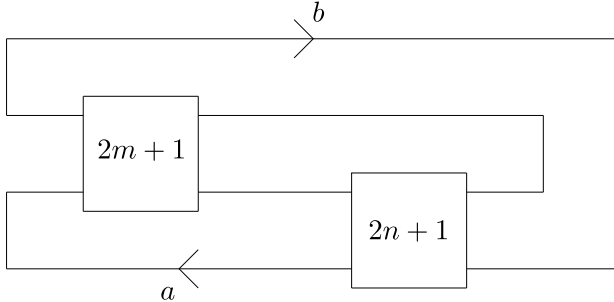


FIGURE 1. The double twist link  $J(2m+1, 2n+1)$ . Here  $2m+1$  and  $2n+1$  denote the numbers of half twists in the boxes. Positive (resp. negative) numbers correspond to right-handed (resp. left handed) twists.

**Theorem 1.1.** *For  $\alpha \in (0, \alpha_{\mathcal{L}_m})$  we have*

$$\text{Vol } E_{\mathcal{L}_m}(\alpha) = \int_{\alpha}^{\pi} \log \left| \frac{S_m(z) - e^{-i\omega} S_{m-1}(z)}{S_m(z) - e^{i\omega} S_{m-1}(z)} \right| d\omega$$

where  $z$ , with  $\text{Im}(S_{m-1}(z)\overline{S_m(z)}) \geq 0$ , is a certain root of  $R_{\mathcal{L}_m}(e^{i\omega/2}, z) = 0$ .

Note that the above volume formula for the hyperbolic cone-manifold  $E_{\mathcal{L}_m}(\alpha)$  depends on the choice of a root  $z$ , with  $\text{Im}(S_{m-1}(z)\overline{S_m(z)}) \geq 0$ , of  $R_{\mathcal{L}_m}(e^{i\omega/2}, z) = 0$ . In numerical approximations, we choose the root  $z$  which gives the maximal volume.

It is known that the volume of the  $k$ -fold cyclic covering over a hyperbolic link  $\mathcal{L}$  is  $k$  times the volume of the hyperbolic cone-manifold of  $\mathcal{L}$  with cone-angle  $2\pi/k$ . As a direct consequence of Theorem 1.1, we obtain the following.

**Corollary 1.2.** *The hyperbolic volume of the  $k$ -fold cyclic covering over the two-bridge link  $\mathcal{L}_m$ , with  $k \geq 3$ , is given by the following formula*

$$k \text{Vol } E_{\mathcal{L}_m}\left(\frac{2\pi}{k}\right) = k \int_{\frac{2\pi}{k}}^{\pi} \log \left| \frac{S_m(z) - e^{-i\omega} S_{m-1}(z)}{S_m(z) - e^{i\omega} S_{m-1}(z)} \right| d\omega$$

where  $z$ , with  $\text{Im}(S_{m-1}(z)\overline{S_m(z)}) \geq 0$ , is a certain root of  $R_{\mathcal{L}_m}(e^{i\omega/2}, z) = 0$ .

The A-polynomial of a knot in  $S^3$  was introduced by Cooper, Culler, Gillet, Long and Shalen [CCGLS] in the 90's. It describes the  $SL_2(\mathbb{C})$ -character variety of the knot complement as viewed from the boundary torus. The A-polynomial carries a lot of information about the topology of the knot. For example, the sides of the Newton polygon of the A-polynomial of a knot in  $S^3$  give rise to incompressible surfaces in the knot complement [CCGLS]. A generalization of the A-polynomial to links in  $S^3$  was proposed by Zhang [Zh]. For an  $\ell$ -component link in  $S^3$ , Zhang defined a polynomial  $\ell$ -tuple link invariant called the A-polynomial  $\ell$ -tuple. The A-polynomial 1-tuple of a knot is just its A-polynomial. The A-polynomial  $\ell$ -tuple also carries important information about the topology of the link. For example, it can be used to construct concrete examples of hyperbolic link manifolds with non-integral traces [Zh].

The A-polynomial 2-tuple has been computed for a family of two bridge links called twisted Whitehead links [Tr]. In this paper we compute the A-polynomial 2-tuple for the canonical component of the character variety of  $\mathcal{L}_m = J(2m+1, 2m+1)$ .

**Theorem 1.3.** *Let  $\{Q_j(s, w)\}_{j \in \mathbb{Z}}$  be the sequence of polynomials in two variables  $s, w$  defined by  $Q_{-1} = Q_0 = 2$  and*

$$Q_j = \alpha Q_{j-1} - Q_{j-2} + \beta$$

where

$$\begin{aligned} \alpha &= (s^8 + s^4)w^4 + (-2s^8 + 6s^6 + 6s^4 - 2s^2)w^3 + (s^8 - 12s^6 + 34s^4 - 12s^2 + 1)w^2 \\ &\quad + (-2s^6 + 6s^4 + 6s^2 - 2)w + s^4 + 1, \\ \beta &= -2(s^2 - 1)^2 (s^4 w^4 - (s^4 + s^2)w^3 - 6s^2 w^2 - (s^2 + 1)w + 1). \end{aligned}$$

Then the A-polynomial 2-tuple corresponding to the canonical component of the character variety of  $\mathcal{L}_m$  is  $[A(M, L), A(M, L)]$  where  $A(M, L) = (L - 1)Q_m(M, LM^{2m})$ .

The paper is organized as follows. In Section 2 we review the definition of the A-polynomial  $\ell$ -tuple of an  $\ell$ -component link in  $S^3$ . In Section 3 we compute the nonabelian  $SL_2(\mathbb{C})$ -representations of the double twist link  $J(2m+1, 2n+1)$ . In Section 4 we compute the volume of hyperbolic cone-manifolds of  $\mathcal{L}_m = J(2m+1, 2m+1)$  and give a proof of Theorem 1.1. The last section is devoted to the computation of the A-polynomial 2-tuple for the canonical component of the character variety of  $\mathcal{L}_m$  and a proof of Theorem 1.3.

## 2. THE A-POLYNOMIAL $\ell$ -TUPLE OF A LINK

**2.0.1. Character varieties.** The set of characters of representations of a finitely generated group  $G$  into  $SL_2(\mathbb{C})$  is known to be a algebraic set over  $\mathbb{C}$  [CS, LM]. It is called the character variety of  $G$  and denoted by  $\chi(G)$ . For example, the character variety  $\chi(\mathbb{Z}^2)$  of the free abelian group on 2 generators  $\mu, \lambda$  is isomorphic to  $(\mathbb{C}^*)^2/\tau$ , where  $(\mathbb{C}^*)^2$  is the set of non-zero complex pairs  $(M, L)$  and  $\tau : (\mathbb{C}^*)^2 \rightarrow (\mathbb{C}^*)^2$  is the involution defined by  $\tau(M, L) = (M^{-1}, L^{-1})$ . This fact can be proved by noting that every representation  $\rho : \mathbb{Z}^2 \rightarrow SL_2(\mathbb{C})$  is conjugate to an upper diagonal one, with  $M$  and  $L$  being the upper left entries of  $\rho(\mu)$  and  $\rho(\lambda)$  respectively.

**2.0.2. The A-polynomial.** Suppose  $\mathcal{L} = K_1 \sqcup \cdots \sqcup K_\ell$  be an  $\ell$ -component link in  $S^3$ . Let  $E_{\mathcal{L}} = S^3 \setminus \mathcal{L}$  be the link exterior and  $T_1, \dots, T_\ell$  the boundary tori of  $E_{\mathcal{L}}$  corresponding to  $K_1, \dots, K_\ell$  respectively. Each  $T_j$  is a torus whose fundamental group is free abelian of rank two. An orientation of  $K_j$  will define a unique pair of an oriented meridian  $\mu_j$  and an oriented longitude  $\lambda_j$  such that the linking number between the longitude  $\lambda_j$  and the knot  $K_j$  is 0. The pair provides an identification of  $\chi(\pi_1(T_j))$  and  $(\mathbb{C}^*)_j^2/\tau_j$ , where  $(\mathbb{C}^*)_j^2$  is the set of non-zero complex pairs  $(M_j, L_j)$  and  $\tau_j$  is the involution  $\tau(M_j, L_j) = (M_j^{-1}, L_j^{-1})$ , which actually does not depend on the orientation of  $K_j$ .

The inclusion  $T_j \hookrightarrow E_{\mathcal{L}}$  induces the restriction map

$$\rho_j : \chi(\pi_1(E_{\mathcal{L}})) \longrightarrow \chi(\pi_1(T_j)) \equiv (\mathbb{C}^*)_j^2/\tau_j.$$

For each  $\gamma \in \pi_1(E_{\mathcal{L}})$  let  $f_\gamma$  be the regular function on  $\chi(\pi_1(E_{\mathcal{L}}))$  defined by

$$f_\gamma(\chi_\rho) = (\chi_\rho(\gamma))^2 - 4 = (\text{tr } \rho(\gamma))^2 - 4,$$

where  $\chi_\rho$  denotes the character of a representation  $\rho : \pi_1(E_{\mathcal{L}}) \rightarrow SL_2(\mathbb{C})$ . Let  $\chi_j(\pi_1(E_{\mathcal{L}}))$  be the subvariety of  $\chi(\pi_1(E_{\mathcal{L}}))$  defined by  $f_{\mu_k} = 0$ ,  $f_{\lambda_k} = 0$  for all  $k \neq j$ . Let  $Z_j$  be the image of  $\chi_j(\pi_1(E_{\mathcal{L}}))$  under  $\rho_j$  and  $\hat{Z}_j \subset (\mathbb{C}^*)_j^2$  the lift of  $Z_j$  under the projection  $(\mathbb{C}^*)_j^2 \rightarrow (\mathbb{C}^*)_j^2/\tau_i$ . It is known that the Zariski closure of  $\hat{Z}_j \subset (\mathbb{C}^*)_j^2 \subset \mathbb{C}_j^2$  in  $\mathbb{C}_j^2$  is an

algebraic set consisting of components of dimension 0 or 1 [Zh]. The union of all the 1-dimension components is defined by a single polynomial  $A_j \in \mathbb{Z}[M_j, L_j]$  whose coefficients are co-prime. Note that  $A_j$  is defined up to  $\pm 1$ . We will call  $[A_1(M_1, L_1), \dots, A_\ell(M_\ell, L_\ell)]$  the A-polynomial  $\ell$ -tuple of  $\mathcal{L}$ . For brevity, we also write  $A_j(M, L)$  for  $A_j(M_j, L_j)$ . We refer the reader to [Zh] for properties of the A-polynomial  $\ell$ -tuple.

### 3. DOUBLE TWIST LINKS $J(2m+1, 2n+1)$

In this section we compute nonabelian  $SL_2(\mathbb{C})$ -representations of the double twist link  $J(2m+1, 2n+1)$ . They are described by the Chebyshev polynomials of the second kind, and so we first recall some properties of these polynomials.

**3.1. Chebyshev polynomials.** Recall that  $\{S_j(v)\}_{j \in \mathbb{Z}}$  is the sequence of the Chebyshev polynomials of the second kind defined by  $S_0(v) = 1$ ,  $S_1(v) = v$  and  $S_j(v) = vS_{j-1}(v) - S_{j-2}(v)$  for all integers  $j$ . The following two lemmas are elementary, see e.g. [Tr].

**Lemma 3.1.** *For any integer  $j$  we have*

$$S_j^2(v) + S_{j-1}^2(v) - vS_j(v)S_{j-1}(v) = 1.$$

**Lemma 3.2.** *Suppose  $V \in SL_2(\mathbb{C})$  and  $v = \text{tr } V$ . For any integer  $j$  we have*

$$V^j = S_j(v)\mathbf{1} - S_{j-1}(v)V^{-1}$$

where  $\mathbf{1}$  denotes the  $2 \times 2$  identity matrix.

We will need the following lemma in the last section of the paper.

**Lemma 3.3.** *For any integer  $j$  we have*

$$S_j(z)S_{j-1}(z) = (z^2 - 2)S_{j-1}(z)S_{j-2}(z) - S_{j-2}(z)S_{j-3}(z) + z.$$

*Proof.* We have  $S_j(z)S_{j-1}(z) + S_{j-2}(z)S_{j-3}(z)$

$$\begin{aligned} &= (zS_{j-1}(z) - S_{j-2}(z))S_{j-1}(z) + S_{j-2}(z)(zS_{j-2}(z) - S_{j-1}(z)) \\ &= z(S_{j-1}^2(z) + S_{j-2}^2(z)) - 2S_{j-1}(z)S_{j-2}(z). \end{aligned}$$

The lemma follows, since  $S_{j-1}^2(z) + S_{j-2}^2(z) = 1 + zS_{j-1}(z)S_{j-2}(z)$  by Lemma 3.1.  $\square$

**3.2. Nonabelian representations.** In this subsection we study representations of link groups into  $SL_2(\mathbb{C})$ . A representation is called nonabelian if its image is a nonabelian subgroup of  $SL_2(\mathbb{C})$ . Let  $\mathcal{L} = J(2m+1, 2n+1)$  and  $E_{\mathcal{L}} = S^3 \setminus \mathcal{L}$  the link exterior. By [PT] (and [MPL] also) the link group of  $\mathcal{L}$  has a two-generator presentation

$$\pi_1(E_{\mathcal{L}}) = \langle a, b \mid aw = wa \rangle,$$

where  $w = (b^{-1}a)^m[(ba^{-1})^mba(b^{-1}a)^m]^n$  and  $a, b$  are meridians depicted in Figure 1.

Suppose  $\rho : \pi_1(E_{\mathcal{L}}) \rightarrow SL_2(\mathbb{C})$  is a nonabelian representation. Up to conjugation, we may assume that

$$(3.1) \quad \rho(a) = \begin{bmatrix} s_1 & 1 \\ 0 & s_1^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} s_2 & 0 \\ u & s_2^{-1} \end{bmatrix}$$

where  $(u, s_1, s_2) \in (\mathbb{C}^*)^3$  satisfies the matrix equation  $\rho(aw) = \rho(wa)$ . For any word  $v$  in 2 letters  $a$  and  $b$ , we write  $\rho(v) = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$ . Then, by Riley [Ri],  $w_{12}$  can be written

as  $w_{21} = uw'_{21}$  for some  $w'_{12} \in \mathbb{C}[s_1^{\pm 1}, s_2^{\pm 1}, u]$  and the matrix equation  $\rho(aw) = \rho(wa)$  is equivalent to the single equation  $w'_{12} = 0$ . We call  $w'_{12}$  the Riley polynomial of  $\mathcal{L}$ .

We now compute  $w'_{12}$  explicitly. Let  $x = \text{tr } \rho(a) = s_1 + s_1^{-1}$ ,  $y = \text{tr } \rho(b) = s_2 + s_2^{-1}$  and  $z = \text{tr } \rho(ab^{-1}) = s_1s_2^{-1} + s_1^{-1}s_2 - u$ .

Let  $c = (b^{-1}a)^m$  and  $d = (ba^{-1})^mba(b^{-1}a)^m = bc^{-1}ac$ . Then  $w = cd^n$ . Since

$$\rho(b^{-1}a) = \begin{bmatrix} s_1s_2^{-1} & s_2^{-1} \\ -s_1u & s_1^{-1}s_2 - u \end{bmatrix},$$

by Lemma 3.2 we have  $\rho(c) = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$  where

$$\begin{aligned} c_{11} &= S_m(z) - (s_1^{-1}s_2 - u)S_{m-1}(z), \\ c_{12} &= s_2^{-1}S_{m-1}(z), \\ c_{21} &= -s_1uS_{m-1}(z), \\ c_{22} &= S_m(z) - s_1s_2^{-1}S_{m-1}(z). \end{aligned}$$

By a direct computation we then have  $\rho(d) = \rho(bc^{-1}ac) = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$  where

$$\begin{aligned} d_{11} &= s_1s_2S_m^2(z) - (s_1^2 + s_2^2)S_m(z)S_{m-1}(z) + (s_1s_2 + u)S_{m-1}^2(z), \\ d_{12} &= s_2S_m^2(z) - (s_1 + s_1^{-1})S_m(z)S_{m-1}(z) + s_2^{-1}S_{m-1}^2(z), \\ d_{21} &= u(s_1S_m^2(z) - (s_2 + s_2^{-1})S_m(z)S_{m-1}(z) + s_1^{-1}S_{m-1}^2(z)), \\ d_{22} &= (s_1^{-1}s_2^{-1} + u)S_m^2(z) - (s_1^{-2} + s_2^{-2})S_m(z)S_{m-1}(z) + s_1^{-1}s_2^{-1}S_{m-1}^2(z). \end{aligned}$$

Let  $t = \text{tr } \rho(d)$ . From the above computations we have

$$\begin{aligned} t &= (s_1s_2 + s_1^{-1}s_2^{-1} + u)(S_m^2(z) + S_{m-1}^2(z)) - (s_1^2 + s_1^{-2} + s_2^2 + s_2^{-2})S_m(z)S_{m-1}(z) \\ &= (xy - z)(S_m^2(z) + S_{m-1}^2(z)) - (x^2 + y^2 - 4)S_m(z)S_{m-1}(z). \end{aligned}$$

Since  $w = cd^n$ , by Lemma 3.2 we have

$$\rho(w) = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} S_n(t) - d_{22}S_{n-1}(t) & d_{12}S_{n-1}(t) \\ d_{21}S_{n-1}(t) & S_n(t) - d_{11}S_{n-1}(t) \end{bmatrix}.$$

With  $\rho(w) = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$  we obtain

$$\begin{aligned} w_{11} &= c_{11}(S_n(t) - d_{22}S_{n-1}(t)) + c_{12}d_{21}S_{n-1}(t), \\ w_{21} &= c_{21}(S_n(t) - d_{22}S_{n-1}(t)) + c_{22}d_{21}S_{n-1}(t). \end{aligned}$$

By direct computations we have  $w_{21} = us_1(S_m(z)S_{n-1}(t) - S_{m-1}(z)S_n(t))$  and

$$\begin{aligned} w_{11} &= -S_{n-1}(t)\{(s_1s_2^{-1} + s_1^{-1}s_2 + s_1^{-1}s_2^{-1} - z)S_m(z) - s_1^{-2}S_{m-1}(z)\} \\ &\quad + S_n(t)(S_m(z) + (s_1s_2^{-1} - z)S_{m-1}(z)). \end{aligned}$$

Hence, the Riley polynomial of  $\mathcal{L} = J(2m+1, 2n+1)$  is

$$w'_{21} = S_m(z)S_{n-1}(t) - S_{m-1}(z)S_n(t).$$

It determines the nonabelian  $SL_2(\mathbb{C})$ -character variety of  $\mathcal{L}$ , which is essentially the set of all nonabelian representations  $\rho : \pi_1(E_{\mathcal{L}}) \rightarrow SL_2(\mathbb{C})$  up to conjugation. Moreover, for any nonabelian representation  $\rho$  of the form (3.1) we have  $\rho(w) = \begin{bmatrix} w_{11} & * \\ 0 & (w_{11})^{-1} \end{bmatrix}$  where

$$(3.2) \quad w_{11} = -S_{n-1}(t) \{ (s_1^{-1}s_2 + s_1^{-1}s_2^{-1})S_m(z) - s_1^{-2}S_{m-1}(z) \} + S_n(t)S_m(z).$$

Let  $\bar{w}$  is the word obtained from  $w$  by exchanging  $a$  and  $b$ , namely

$$\bar{w} = (a^{-1}b)^m [(ab^{-1})^m ab(a^{-1}b)^m]^n.$$

It is easy to see that the equation  $aw = wa$  is equivalent to  $\bar{w}b = b\bar{w}$ . Moreover, for any nonabelian representation  $\rho$  of the form (3.1) we have  $\rho(\bar{w}) = \begin{bmatrix} \bar{w}_{11} & 0 \\ * & (\bar{w}_{11})^{-1} \end{bmatrix}$  where

$$(3.3) \quad \bar{w}_{11} = -S_{n-1}(t) \{ (s_1s_2^{-1} + s_1^{-1}s_2^{-1})S_m(z) - s_2^{-2}S_{m-1}(z) \} + S_n(t)S_m(z).$$

**Remark 3.4.** The above formula for the nonabelian  $SL_2(\mathbb{C})$ -character variety of the double twist link  $\mathcal{L} = J(2m+1, 2n+1)$  was already obtained in [PT] by a different method. Moreover, it was also shown in [PT] that the nonabelian character variety of  $\mathcal{L}$  is reducible if and only if  $m = n$ . In which case, it has exactly 2 irreducible components and the canonical component is determined by the equation  $t = z$ .

From now on we consider only the double twist link  $\mathcal{L}_m = J(2m+1, 2m+1)$ , where  $m \neq -1, 0$ . As mentioned above, the canonical component of the character variety of  $\mathcal{L}_m$  is given by the equation  $t = z$  where

$$(3.4) \quad t = (xy - z)(S_m^2(z) + S_{m-1}^2(z)) - (x^2 + y^2 - 4)S_m(z)S_{m-1}(z).$$

#### 4. VOLUME OF HYPERBOLIC CONE-MANIFOLDS OF $\mathcal{L}_m$

Recall that  $E_{\mathcal{L}_m}(\alpha)$  is the cone-manifold of  $\mathcal{L}_m$  with cone angles  $\alpha_1 = \alpha_2 = \alpha$ . There exists an angle  $\alpha_{\mathcal{L}_m} \in [\frac{2\pi}{3}, \pi)$  such that  $E_{\mathcal{L}_m}(\alpha)$  is hyperbolic for  $\alpha \in (0, \alpha_{\mathcal{L}_m})$ , Euclidean for  $\alpha = \alpha_{\mathcal{L}_m}$ , and spherical for  $\alpha \in (\alpha_{\mathcal{L}_m}, \pi)$ .

For  $\alpha \in (0, \alpha_{\mathcal{L}_m})$ , by the Schläfli formula we have

$$\text{Vol } E_{\mathcal{L}_m}(\alpha) = \int_{\alpha}^{\pi} 2 \log |w_{11}| d\omega$$

where  $w_{11}$  is the  $(1, 1)$ -entry of the matrix  $\rho(w)$  and  $\rho : \pi_1(\mathcal{L}_m) \rightarrow SL_2(\mathbb{C})$  is a representation of the form (3.1) such that the following 3 conditions hold:

- (i)  $s_1 = s_2 = s = e^{i\omega/2}$ ,
- (ii) the character  $\chi_{\rho}$  of  $\rho$  lies on the canonical component of the character variety of  $\mathcal{L}_m$ ,
- (iii)  $|w_{11}| \geq 1$ .

We refer the reader to [HLM, HLMR] and references therein for the volume formula of hyperbolic cone-manifolds of links using the Schläfli formula.

We now simplify  $w_{11}$  for representations  $\rho$  of the form (3.1) satisfying the conditions (i)–(iii). Consider the canonical component  $t = z$  of the character variety of  $\mathcal{L}_m$ . With  $s_1 = s_2 = s = e^{i\omega/2}$ , equation (3.2) implies that

$$\begin{aligned} w_{11} &= -S_{m-1}(z) \{ (1 + s^{-2})S_m(z) - s^{-2}S_{m-1}(z) \} + S_m^2(z) \\ &= (S_m(z) - S_{m-1}(z))(S_m(z) - s^{-2}S_{m-1}(z)). \end{aligned}$$

Moreover, the equation  $t = z$  can be written as

$$(s^2 + s^{-2} + 2 - z)(S_m^2(z) + S_{m-1}^2(z)) - 2(s^2 + s^{-2})S_m(z)S_{m-1}(z) = z.$$

This, together with  $S_m^2(z) + S_{m-1}^2(z) = 1 + zS_m(z)S_{m-1}(z)$  (by Lemma 3.2), implies that

$$\begin{aligned} S_m(z)S_{m-1}(z) &= \frac{2z - (s^2 + s^{-2} + 2)}{(z - 2)(s^2 + s^{-2} - z)}, \\ S_m^2(z) + S_{m-1}^2(z) &= \frac{z^2 - 2(s^2 + s^{-2})}{(z - 2)(s^2 + s^{-2} - z)}. \end{aligned}$$

Then  $(S_m(z) - S_{m-1}(z))^2 = S_m^2(z) + S_{m-1}^2(z) - 2S_m(z)S_{m-1}(z) = \frac{z-2}{s^2+s^{-2}-z}$  and

$$\begin{aligned} (S_m(z) - s^2 S_{m-1}(z))(S_m(z) - s^{-2} S_{m-1}(z)) &= S_m^2(z) + S_{m-1}^2(z) - (s^2 + s^{-2})S_m(z)S_{m-1}(z) \\ &= \frac{s^2 + s^{-2} - z}{z - 2}. \end{aligned}$$

It follows that  $(S_m(z) - S_{m-1}(z))^2 (S_m(z) - s^2 S_{m-1}(z)) (S_m(z) - s^{-2} S_{m-1}(z)) = 1$  and

$$w_{11}^2 = (S_m(z) - S_{m-1}(z))^2 (S_m(z) - s^{-2} S_{m-1}(z))^2 = \frac{S_m(z) - s^{-2} S_{m-1}(z)}{S_m(z) - s^2 S_{m-1}(z)}.$$

Note that  $|S_m(z) - e^{-i\omega} S_{m-1}(z)| \geq |S_m(z) - e^{i\omega} S_{m-1}(z)|$  if and only if  $\text{Im}(S_{m-1}(z)\overline{S_m(z)}) \geq 0$ . Hence, for  $\alpha \in (0, \alpha_{\mathcal{L}_m})$ , by the Schlaflf formula we have

$$\text{Vol } E_{\mathcal{L}_m}(\alpha) = \int_{\alpha}^{\pi} 2 \log |w_{11}| d\omega = \int_{\alpha}^{\pi} \log \left| \frac{S_m(z) - s^{-2} S_{m-1}(z)}{S_m(z) - s^2 S_{m-1}(z)} \right| d\omega$$

where  $s = e^{i\omega/2}$  and  $z$ , with  $\text{Im}(S_{m-1}(z)\overline{S_m(z)}) \geq 0$ , satisfy

$$(s^2 + s^{-2} + 2 - z)(S_m^2(z) + S_{m-1}^2(z)) - 2(s^2 + s^{-2})S_m(z)S_{m-1}(z) - z = 0.$$

This completes the proof of Theorem 1.1.

## 5. THE A-POLYNOMIAL 2-TUPLE OF $\mathcal{L}_m$

The canonical longitudes corresponding to the meridians  $a$  and  $b$  of  $J(2m+1, 2n+1)$  are respectively  $\lambda_a = wa^{-2n}$  and  $\lambda_b = \overline{w}b^{-2n}$ , where  $\overline{w} = (a^{-1}b)^m [(ab^{-1})^m ab(a^{-1}b)^m]^n$  is the word obtained from  $w$  by exchanging  $a$  and  $b$ .

Consider the canonical component  $t = z$  of the character variety of  $\mathcal{L}_m = J(2m+1, 2m+1)$ . To compute the A-polynomial 2-tuple for this component, we first consider a representation  $\rho : \pi_1(\mathcal{L}_m) \rightarrow SL_2(\mathbb{C})$  of the form (3.1) and find a polynomial relating  $s_1$  and  $w_{11}$  when both  $t = z$  and  $s_2^2 = (\overline{w}_{11})^2 = 1$  occur. Recall from Subsection 3.2 that  $w_{11}$  and  $\overline{w}_{11}$  are upper left entries of  $\rho(w)$  and  $\rho(\overline{w})$  respectively.

With  $t = z$  and  $s_2 = 1$ , by equations (3.2) and (3.3) we have

$$\begin{aligned} w_{11} &= -S_{m-1}(z) \{2s_1^{-1}S_m(z) - s_1^{-2}S_{m-1}(z)\} + S_m^2(z) \\ &= (S_m(z) - s_1^{-1}S_{m-1}(z))^2 \end{aligned}$$

and

$$\begin{aligned} \overline{w}_{11} &= -S_{m-1}(z) \{(s_1 + s_1^{-1})S_m(z) - S_{m-1}(z)\} + S_m^2(z). \\ &= (S_m(z) - s_1 S_{m-1}(z))(S_m(z) - s_1^{-1}S_{m-1}(z)). \end{aligned}$$

Moreover, since  $S_m^2(z) + S_{m-1}^2(z) = 1 + zS_m(z)S_{m-1}(z)$ , the equation  $t = z$  becomes

$$\begin{aligned} 0 &= (2x - z)(S_m^2(z) + S_{m-1}^2(z)) - x^2S_m(z)S_{m-1}(z) - z \\ &= (2x - z)(1 + zS_m(z)S_{m-1}(z)) - x^2S_m(z)S_{m-1}(z) - z \\ &= (x - z)(2 + (z - x)S_m(z)S_{m-1}(z)). \end{aligned}$$

Suppose  $z - x = 0$ . Then  $\overline{w}_{11} = -S_{m-1}(z)\{zS_m(z) - S_{m-1}(z)\} + S_m^2(z) = 1$  and

$$w_{11} = (S_m(x) - s^{-1}S_{m-1}(x))^2 = s^{2m}.$$

Here we use the fact that  $S_j(s_1 + s_1^{-1}) = (s_1^{j+1} - s_1^{-j-1})/(s_1 - s_1^{-1})$  for all integers  $j$ .

Suppose  $2 + (z - x)S_m(z)S_{m-1}(z) = 0$ . This is equivalent to

$$(5.1) \quad (S_m(z) - s_1S_{m-1}(z))(S_m(z) - s_1^{-1}S_{m-1}(z)) = -1,$$

since  $S_m^2(z) + S_{m-1}^2(z) = 1 + zS_m(z)S_{m-1}(z)$ . It follows that  $\overline{w}_{11} = -1$  and

$$w_{11} = (S_m(z) - s_1^{-1}S_{m-1}(z))^2 = -\frac{S_m(z) - s_1^{-1}S_{m-1}(z)}{S_m(z) - s_1S_{m-1}(z)}.$$

Hence  $S_m(z) = rS_{m-1}(z)$  where  $r = \frac{s_1w_{11} + s_1^{-1}}{w_{11} + 1}$ . We have

$$1 = S_m^2(z) + S_{m-1}^2(z) - zS_m(z)S_{m-1}(z) = S_{m-1}^2(z)(1 - zr + r^2),$$

which implies that  $S_{m-1}^2(z) = (1 - zr + r^2)^{-1}$ . Equation (5.1) then becomes

$$-1 = S_{m-1}^2(z)(r - s_1)(r - s_1^{-1}) = (r - s_1)(r - s_1^{-1})/(1 - zr + r^2).$$

By solving for  $z$  from the above equation, we obtain

$$z = 2 \left( r + \frac{1}{r} \right) - (s_1 + s_1^{-1}) = 2 \left( \frac{s_1w_{11} + s_1^{-1}}{w_{11} + 1} + \frac{w_{11} + 1}{s_1w_{11} + s_1^{-1}} \right) - (s_1 + s_1^{-1}).$$

Now, by plugging this expression of  $z$  into the equation  $2 + (z - x)S_m(z)S_{m-1}(z) = 0$  we obtain a polynomial (depending on  $m$ ) relating  $s_1$  and  $w_{11}$ . Moreover, we can find a recurrence relation between these polynomials as follows.

Let  $P_m(x, z) = 2 + (z - x)S_m(z)S_{m-1}(z)$ . By Lemma 3.3 we have  $S_m(z)S_{m-1}(z) = (z^2 - 2)S_{m-1}(z)S_{m-2}(z) - S_{m-2}(z)S_{m-3}(z) + z$ . This implies that

$$\begin{aligned} P_m &= 2 + (z^2 - 2)(P_{m-1} - 2) - (P_{m-2} - 2) + z(z - x) \\ &= (z^2 - 2)P_{m-1} - P_{m-2} + 8 - z(z + x). \end{aligned}$$

Let  $Q_m(s_1, w_{11}) = s_1^2(w_{11} + 1)^2(s_1^2w_{11} + 1)^2P_m(x, z)$ . By replacing

$$z = 2 \left( \frac{s_1w_{11} + s_1^{-1}}{w_{11} + 1} + \frac{w_{11} + 1}{s_1w_{11} + s_1^{-1}} \right) - (s_1 + s_1^{-1})$$

into the above recurrence relation for  $P_m$  we have

$$Q_m = \alpha Q_{m-1} - Q_{m-2} + \beta$$

where

$$\begin{aligned} \alpha &= (s_1^8 + s_1^4)w_{11}^4 + (-2s_1^8 + 6s_1^6 + 6s_1^4 - 2s_1^2)w_{11}^3 + (s_1^8 - 12s_1^6 + 34s_1^4 - 12s_1^2 + 1)w_{11}^2 \\ &\quad + (-2s_1^6 + 6s_1^4 + 6s_1^2 - 2)w_{11} + s_1^4 + 1, \\ \beta &= -2(s_1^2 - 1)^2(s_1^4w_{11}^4 - (s_1^4 + s_1^2)w_{11}^3 - 6s_1^2w_{11}^2 - (s_1^2 + 1)w_{11} + 1). \end{aligned}$$



We have shown that  $(\overline{w}_{11})^2 = 1$  and  $(w_{11} - s_1^{2m})Q(s_1, w_{11}) = 0$  when both  $t = z$  and  $s_2 = 1$  occur. The same holds true when both  $t = z$  and  $s_2 = -1$  occur. This implies that  $(w_{11} - s_1^{2m})Q(s_1, w_{11}) = 0$  when both  $t = z$  and  $s_2^2 = (\overline{w}_{11})^2 = 1$  occur.

Similarly, we have  $(\overline{w}_{11} - s_2^{2m})Q(s_2, \overline{w}_{11}) = 0$  when both  $t = z$  and  $s_1^2 = (w_{11})^2 = 1$  occur. Since the canonical longitudes corresponding to the meridians  $a$  and  $b$  of  $\mathcal{L}_m = J(2m + 1, 2m + 1)$  are respectively  $\lambda_a = wa^{-2m}$  and  $\lambda_b = \overline{w}b^{-2m}$ , we conclude that the  $A$ -polynomial 2-tuple corresponding to the canonical component of the character variety of  $\mathcal{L}_m$  is  $[A(M, L), A(M, L)]$  where  $A(M, L) = (L - 1)Q_m(M, LM^{2m})$ .

This completes the proof of Theorem 1.3.

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